

# Uniqueness of complete maximal hypersurfaces in spatially open $(n + 1)$ -dimensional Robertson-Walker spacetimes with flat fiber

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Dedicated to Professor Ángel Ferrández

## Abstract

In this paper, under natural geometric and physical assumptions we provide new uniqueness and non-existence results for complete maximal hypersurfaces in spatially open Robertson-Walker spacetimes whose fiber is flat. Moreover, our results are applied to relevant spacetimes as the steady state spacetime, Einstein-de Sitter spacetime and radiation models.

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## 1 Introduction

The importance in General Relativity of maximal and constant mean curvature spacelike hypersurfaces in spacetimes is well-known; a summary of several reasons justifying it can be found in [13]. Each maximal hypersurface can describe, in some relevant cases, the transition between the expanding and contracting phases of a relativistic universe. Moreover, the existence of constant mean curvature (and in particular maximal) hypersurfaces is necessary for the study of the structure of singularities in the space of solutions to the Einstein equations. Also, the deep understanding of

this kind of hypersurfaces is essential to prove the positivity of the gravitational mass. They are also interesting for Numerical Relativity, where maximal hypersurfaces are used for integrating forward in time. All these physical aspects can be found in [11], [13] and references therein.

A maximal hypersurface is (locally) a critical point for a natural variational problem, namely of the area functional (see, for instance, [5]). From a mathematical point of view, it is necessary to study the maximal hypersurfaces of a spacetime in order to understand its structure [3]. Especially, for some asymptotically flat spacetimes, the existence of a foliation by maximal hypersurfaces is established (see, for instance, [4] and references therein). The existence results and, consequently, uniqueness appear as kernel topics. Classical papers dealing with uniqueness results for constant mean curvature (CMC) hypersurfaces are [4], [6] and [13], although a previous relevant result in this direction was the proof of the Bernstein-Calabi conjecture [7] for the  $n$ -dimensional Lorentz-Minkowski spacetime given by Cheng and Yau [9]. In [4], Brill and Flaherty replaced the Lorentz-Minkowski spacetime by a spatially closed universe, and proved uniqueness results for CMC hypersurfaces in the large by assuming  $\overline{\text{Ric}}(z, z) > 0$  for every timelike vector  $z$ . In [13], this energy condition was relaxed by Marsden and Tipler to include, for instance, non-flat vacuum spacetimes. More recently, Bartnik proved in [3] very general existence theorems and consequently averred that it would be useful to find new satisfactory uniqueness results. Still more recently, in [1] Alías, Romero and Sánchez proved new uniqueness results in the class of spacetimes that they called spatially closed Generalized Robertson-Walker (GRW) spacetimes (which includes the spatially closed Robertson-Walker spacetimes) under the Timelike Convergence Condition. This GRW spacetimes differ from the classical Robertson-Walker spacetimes due to the fact that, despite being both defined as the warped product of an open interval endowed with negative definite metric and a Riemannian manifold as a fiber, they do not necessarily have constant sectional curvature. Finally, Romero, Rubio and Salamanca provided uniqueness results, in the maximal case, for spatially parabolic GRW spacetimes in [18], which are spatially open models whose fiber is a parabolic Riemannian manifold. Moreover, making use of a well-known generalized maximum principle, the same authors obtain in [19] new uniqueness results in other relevant spatially open GRW spacetimes for complete maximal hypersurfaces which are between two spacelike slices (time bounded) and have a bounded hyperbolic angle.

In this paper we focus on the problems of uniqueness and non-existence of complete maximal hypersurfaces immersed in a spatially open Robertson-Walker spacetime with flat fiber. Note that these models have aroused a great deal of interest, since recent observations have shown that the current universe is very close to a spatially flat geometry [8]. This is actually a natural result from inflation in the early universe [12]. We will give results that can be used when the fiber is  $\mathbb{R}^n$ , which is not parabolic for  $n \geq 3$  and therefore, cannot be studied in arbitrary dimension using previous methods. What is more important, we will not need the hyperbolic angle of the hypersurface to be bounded, which was a restrictive assumption used in previous works studying the spatially open case. Since we are not imposing this restriction, we are able to deal with spacelike hypersurfaces approaching the null  $\text{Scri}$  boundary at infinity, such as hyperboloids in Minkowski spacetime.

Our paper is organized as follows. Section 2 is devoted to introduce the basic notation used to describe spacelike hypersurfaces in GRW spacetimes. In Section 3 we provide an inequality involving the hyperbolic angle of a maximal hypersurface immersed in a GRW spacetime whose fiber is Ricci-flat and obeys the Null Convergence Condition (see Lemma 1). This inequality will play a crucial role in our results. In Section 4 we obtain a uniqueness result for complete maximal hypersurfaces (Theorem 8). In order to obtain it, the fundamental tool will be a Liouville-type theorem applied to the inequality obtained in Lemma 1. Finally, in Subsection 4.1 we give several

interpretations of our mathematical assumptions, their physical meaning and their compatibility with the usual Energy Conditions using a perfect fluid model.

## 2 Preliminaries

Let  $(F, g_F)$  be an  $n(\geq 2)$ -dimensional (connected) Riemannian manifold,  $I$  an open interval in  $\mathbb{R}$  endowed with the metric  $-dt^2$  and  $f$  a positive smooth function defined on  $I$ . Then, the product manifold  $I \times F$  endowed with the Lorentzian metric

$$\bar{g} = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_F^*(g_F), \quad (1)$$

where  $\pi_I$  and  $\pi_F$  denote the projections onto  $I$  and  $F$ , respectively, is called a *GRW spacetime* with *fiber*  $(F, g_F)$ , *base*  $(I, -dt^2)$  and *warping function*  $f$ . If the fiber has constant sectional curvature, it is called a *Robertson-Walker spacetime*.

In any GRW spacetime  $\bar{M} = I \times_f F$ , the coordinate vector field  $\partial_t := \partial/\partial t$  is (unitary) timelike, and hence  $\bar{M}$  is time-orientable. The vector field  $\partial_t$  plays a key role in the study of these spacetimes, since it constitutes a proper time synchronizable reference frame, which is geodesic, spatially conformal and irrotational [21]. On the other hand, if we consider the timelike vector field  $K := f(\pi_I) \partial_t$ , from the relation between the Levi-Civita connection of  $\bar{M}$  and those of the base and the fiber [15, Cor. 7.35], it follows that

$$\bar{\nabla}_X K = f'(\pi_I) X \quad (2)$$

for any  $X \in \mathfrak{X}(\bar{M})$ , where  $\bar{\nabla}$  is the Levi-Civita connection of the Lorentzian metric (1). Thus,  $K$  is conformal and its metrically equivalent 1-form is closed.

From (2) we easily see that the divergence on  $\bar{M}$  of the reference frame  $\partial_t$  satisfies  $\text{div}(\partial_t) = n \frac{f'(t)}{f(t)}$ . Therefore, the observers in  $\partial_t$  are spreading out (resp. coming together) if  $f' > 0$  (resp.  $f' < 0$ ).

Given an  $n$ -dimensional manifold  $M$ , an immersion  $\psi : M \rightarrow \bar{M}$  is said to be *spacelike* if the Lorentzian metric (1) induces, via  $\psi$ , a Riemannian metric  $g_M$  on  $M$ . In this case,  $M$  is called a spacelike hypersurface. We will denote by  $\tau := \pi_I \circ \psi$  the restriction of  $\pi_I$  along  $\psi$ .

The time-orientation of  $\bar{M}$  allows to take, for each spacelike hypersurface  $M$  in  $\bar{M}$ , a unique unitary timelike vector field  $N \in \mathfrak{X}^\perp(M)$  globally defined on  $M$  with the same time-orientation as  $\partial_t$ , i.e., such that  $\bar{g}(N, \partial_t) \leq -1$  and  $\bar{g}(N, \partial_t) = -1$  at a point  $p \in M$  if and only if  $N = \partial_t$  at  $p$ . We will denote by  $A$  the shape operator associated to  $N$ . Then, the *mean curvature function* associated to  $N$  is given by  $H := -(1/n)\text{trace}(A)$ . As it is well-known, the mean curvature is constant if and only if the spacelike hypersurface is, locally, a critical point of the  $n$ -dimensional area functional for compactly supported normal variations, under certain constraints of the volume. When the mean curvature vanishes identically, the spacelike hypersurface is called a *maximal* hypersurface.

For a spacelike hypersurface  $\psi : M \rightarrow \bar{M}$  with Gauss map  $N$ , the *hyperbolic angle*  $\varphi$ , at any point of  $M$ , between the unit timelike vectors  $N$  and  $\partial_t$ , is given by  $\cosh \varphi = -\bar{g}(N, \partial_t)$ . For simplicity, throughout this paper we will refer to  $\varphi$  as the *hyperbolic angle function* on  $M$ .

In any GRW spacetime  $\bar{M} = I \times_f F$  there is a remarkable family of spacelike hypersurfaces, namely its spacelike slices  $\{t_0\} \times F$ ,  $t_0 \in I$ . It can be easily seen that a spacelike hypersurface in

$\overline{M}$  is a (piece of) spacelike slice if and only if the function  $\tau$  is constant. Furthermore, a spacelike hypersurface in  $\overline{M}$  is a (piece of) spacelike slice if and only if the hyperbolic angle  $\varphi$  vanishes identically. The shape operator of the spacelike slice  $\tau = t_0$  is given by  $A = -f'(t_0)/f(t_0)\mathbb{I}$ , where  $\mathbb{I}$  denotes the identity transformation, and therefore its (constant) mean curvature is  $H = f'(t_0)/f(t_0)$ . Thus, a spacelike slice is maximal if and only if  $f'(t_0) = 0$  (and hence, totally geodesic).

### 3 Set up

Let  $\psi : M \rightarrow \overline{M}$  be an  $n$ -dimensional spacelike hypersurface immersed in a GRW spacetime  $\overline{M} = I \times_f F$ . If we denote by

$$\partial_t^T := \partial_t + \overline{g}(N, \partial_t)N$$

the tangential component of  $\partial_t$  along  $\psi$ , then it is easy to check that the gradient of  $\tau$  on  $M$  is

$$\nabla \tau = -\partial_t^T \quad (3)$$

and so

$$|\nabla \tau|^2 = g_M(\nabla \tau, \nabla \tau) = \sinh^2 \varphi. \quad (4)$$

Moreover, since the tangential component of  $K$  along  $\psi$  is given by  $K^T = K + \overline{g}(K, N)N$ , a direct computation from (2) gives

$$\nabla \overline{g}(K, N) = -AK^T \quad (5)$$

where we have used (3), and also

$$\nabla \cosh \varphi = A\partial_t^T - \frac{f'(\tau)}{f(\tau)}\overline{g}(N, \partial_t)\partial_t^T.$$

On the other hand, if we represent by  $\nabla$  the Levi-Civita connection of the metric  $g_M$ , then the Gauss and Weingarten formulas for the immersion  $\psi$  are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - g_M(AX, Y)N \quad (6)$$

and

$$AX = -\overline{\nabla}_X N, \quad (7)$$

where  $X, Y \in \mathfrak{X}(M)$ . Then, taking the tangential component in (2) and using (6) and (7), we get

$$\nabla_X K^T = -f(\tau)\overline{g}(N, \partial_t)AX + f'(\tau)X \quad (8)$$

where  $X \in \mathfrak{X}(M)$  and  $f'(\tau) := f' \circ \tau$ . Since also  $K^T = f(\tau)\partial_t^T$ , it follows from (3) and (8) that the Laplacian of  $\tau$  on  $M$  is

$$\Delta \tau = -\frac{f'(\tau)}{f(\tau)}\{n + |\nabla \tau|^2\} - nH\overline{g}(N, \partial_t). \quad (9)$$

Consequently,

$$\begin{aligned} \Delta f(\tau) &= f'(\tau)\Delta \tau + f''(\tau)|\nabla \tau|^2 \\ &= -\frac{f'(\tau)^2}{f(\tau)}n + |\nabla \tau|^2 f(\tau)(\log f)''(\tau) + nHf'(\tau)\cosh \varphi \end{aligned} \quad (10)$$

and so

$$\begin{aligned}
\Delta(f(\tau) \cosh \varphi) &= \cosh \varphi \Delta f(\tau) + f(\tau) \Delta \cosh \varphi + 2g_M(\nabla f(\tau), \nabla \cosh \varphi) \\
&= -\frac{f'(\tau)^2}{f(\tau)} n \cosh \varphi + f(\tau) \cosh \varphi \sinh^2 \varphi (\log f)''(\tau) + nHf'(\tau) \cosh^2 \varphi \\
&\quad + f(\tau) \Delta \cosh \varphi - 2f'(\tau)g_M(A\partial_t^T, \partial_t^T) - 2\frac{f'(\tau)^2}{f(\tau)} \cosh \varphi \sinh^2 \varphi, \quad (11)
\end{aligned}$$

where we have used (3)-(5).

Furthermore, if we assume that  $M$  is a maximal hypersurface, we get from the Codazzi equation for  $M$  that (see [1, Eq. 8])

$$\Delta(f(\tau) \cosh \varphi) = -\Delta \bar{g}(K, N) = -\overline{\text{Ric}}(K^T, N) + f(\tau) \cosh \varphi \text{trace}(A^2) \quad (12)$$

where  $\overline{\text{Ric}}$  stands for the Ricci tensor on  $\bar{M}$ . Therefore, from (11) and (12) we have

$$\begin{aligned}
\overline{\text{Ric}}(K^T, N) &= f(\tau) \cosh \varphi \text{trace}(A^2) + \frac{f'(\tau)^2}{f(\tau)} n \cosh \varphi \\
&\quad - f(\tau) \cosh \varphi \sinh^2 \varphi (\log f)''(\tau) - f(\tau) \Delta \cosh \varphi \\
&\quad + 2f'(\tau)g_M(A\partial_t^T, \partial_t^T) + 2\frac{f'(\tau)^2}{f(\tau)} \cosh \varphi \sinh^2 \varphi. \quad (13)
\end{aligned}$$

If we put  $N = N_F - \bar{g}(N, \partial_t)\partial_t$ , where  $N_F$  denotes the projection of  $N$  on the fiber  $F$ , it is easy to obtain from (1) that

$$\sinh^2 \varphi = f(\tau)^2 g_F(N_F, N_F). \quad (14)$$

Besides, from [15, Cor. 7.43] we know that

$$\overline{\text{Ric}}(\partial_t, \partial_t) = -n \frac{f''(\tau)}{f(\tau)} \quad (15)$$

and

$$\overline{\text{Ric}}(N_F, N_F) = \sinh^2 \varphi \left( \frac{f''(\tau)}{f(\tau)} + (n-1) \frac{f'(\tau)^2}{f(\tau)^2} \right) \quad (16)$$

where we have used (14) and the fact that  $F$  is Ricci-flat. Then, from (15) and (16) we obtain

$$\begin{aligned}
\overline{\text{Ric}}(K^T, N) &= -f(\tau) \cosh \varphi \overline{\text{Ric}}(N_F, N_F) - f(\tau) \cosh \varphi \sinh^2 \varphi \overline{\text{Ric}}(\partial_t, \partial_t) \\
&= (n-1)f(\tau) \cosh \varphi \sinh^2 \varphi (\log f)''(\tau). \quad (17)
\end{aligned}$$

Finally, from (13) and (17) we get

$$\begin{aligned}
\Delta \cosh \varphi &= -n \cosh \varphi \sinh^2 \varphi (\log f)''(\tau) + \frac{f'(\tau)^2}{f(\tau)^2} \cosh \varphi (n + 2 \sinh^2 \varphi) \\
&\quad + \cosh \varphi \text{trace}(A^2) + 2\frac{f'(\tau)}{f(\tau)} g_M(A\partial_t^T, \partial_t^T). \quad (18)
\end{aligned}$$

On the other hand, the square algebraic trace-norm of the Hessian tensor of  $\tau$  is just

$$|\text{Hess}(\tau)|^2 = \text{trace}(H_\tau \circ H_\tau),$$

where  $H_\tau$  denotes the operator defined by  $g_M(H_\tau(X), Y) := \text{Hess}(\tau)(X, Y)$  for all  $X, Y \in \mathfrak{X}(M)$ .

Taking the tangential component in (2) and using (3) we get that

$$\begin{aligned} |\text{Hess}(\tau)|^2 &= \frac{f'(\tau)^2}{f(\tau)^2} (n-1 + \cosh^4 \varphi) + \cosh^2 \varphi \text{trace}(A^2) \\ &\quad + 2 \frac{f'(\tau)}{f(\tau)} \cosh \varphi g_M(A \partial_t^T, \partial_t^T). \end{aligned} \quad (19)$$

Since  $|\text{Hess}(\tau)|^2 \geq 0$ , it is a straightforward computation to obtain, making use of (18) and (19), that

$$\begin{aligned} \cosh \varphi \Delta \cosh \varphi &\geq -n \cosh^2 \varphi \sinh^2 \varphi (\log f)''(\tau) + n \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi \\ &\quad + 2 \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi \sinh^2 \varphi - \frac{f'(\tau)^2}{f(\tau)^2} (n-1 + \cosh^4 \varphi). \end{aligned} \quad (20)$$

Now, from (20) we have

$$\begin{aligned} \cosh \varphi \Delta \cosh \varphi &\geq -n \frac{f''(\tau)}{f(\tau)} \cosh^2 \varphi \sinh^2 \varphi + n \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi \sinh^2 \varphi + n \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi \\ &\quad + 2 \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi \sinh^2 \varphi - (n-1) \frac{f'(\tau)^2}{f(\tau)^2} - \frac{f'(\tau)^2}{f(\tau)^2} (\sinh^2 \varphi + 1) \cosh^2 \varphi \\ &= -n \frac{f''(\tau)}{f(\tau)} \cosh^2 \varphi \sinh^2 \varphi + (n-1) \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi \sinh^2 \varphi \\ &\quad + 2 \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi \sinh^2 \varphi + (n-1) \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi - (n-1) \frac{f'(\tau)^2}{f(\tau)^2}. \end{aligned} \quad (21)$$

Since  $\cosh^2 \varphi \geq 1$ , we obtain from (21)

$$\cosh \varphi \Delta \cosh \varphi \geq -n \frac{f''(\tau)}{f(\tau)} \cosh^2 \varphi \sinh^2 \varphi + (n+1) \frac{f'(\tau)^2}{f(\tau)^2} \cosh^2 \varphi \sinh^2 \varphi. \quad (22)$$

Now, we introduce an extra assumption with physical meaning. A spacetime  $\overline{M}$  obeys the *Null Convergence Condition* if its Ricci tensor  $\overline{\text{Ric}}$  satisfies  $\overline{\text{Ric}}(z, z) \geq 0$ , for all null vectors  $z$ . When  $\overline{M}$  is a GRW spacetime  $I \times_f F$  with Ricci-flat fiber, it is not difficult to see that this energy condition is satisfied if and only if  $(\log f)''(t) \leq 0$ . Using this assumption, in (22) we have

$$\cosh \varphi \Delta \cosh \varphi \geq \left( (n+1) \frac{f'(\tau)^2}{f(\tau)^2} - n \frac{f''(\tau)}{f(\tau)} \right) \sinh^4 \varphi. \quad (23)$$

Moreover, we know that

$$\frac{1}{2}\Delta \sinh^2 \varphi = \cosh \varphi \Delta \cosh \varphi + |\nabla \cosh \varphi|^2 \geq \cosh \varphi \Delta \cosh \varphi. \quad (24)$$

From (23) and (24) we obtain the following result

**Lemma 1** *Let  $\psi : M \rightarrow \overline{M}$  be an  $n$ -dimensional maximal hypersurface immersed in a GRW spacetime  $\overline{M} = I \times_f F$  with Ricci-flat fiber that obeys the Null Convergence Condition, then*

$$\frac{1}{2}\Delta \sinh^2 \varphi \geq \left( (n+1) \frac{f'(\tau)^2}{f(\tau)^2} - n \frac{f''(\tau)}{f(\tau)} \right) \sinh^4 \varphi. \quad (25)$$

This inequality suggests us the use of the following lemma given by Nishikawa in [14], which extends and clarifies a technical step in Cheng and Yau's seminal paper [9]. In that paper, they used the lemma to study  $\text{trace}(A^2)$ , whereas we will use it on the function  $\sinh^2 \varphi$ .

**Lemma 2** [14] *Let  $M$  be a complete Riemannian manifold whose Ricci curvature is bounded from below and let  $u : M \rightarrow \mathbb{R}$  be a non-negative smooth function on  $M$ . If there exists a constant  $c > 0$  such that  $\Delta u \geq cu^2$ , then  $u$  vanishes identically on  $M$ .*

**Lemma 3** *Let  $\psi : M \rightarrow \overline{M}$  be an  $n$ -dimensional maximal hypersurface immersed in a Robertson-Walker spacetime  $\overline{M} = I \times_f F$  with flat fiber that obeys the Null Convergence Condition. Then, the Ricci curvature of  $M$  must be non-negative.*

*Proof.* Given  $p \in M$ , let us take a local orthonormal frame  $\{U_1, \dots, U_n\}$  around  $p$ . From the Gauss equation we get that the Ricci curvature of  $M$ ,  $\text{Ric}$ , satisfies

$$\text{Ric}(Y, Y) \geq \sum_k \bar{g}(\bar{\mathbf{R}}(Y, U_k)U_k, Y),$$

for all  $Y \in \mathfrak{X}(M)$ , where  $\bar{\mathbf{R}}$  denotes the curvature tensor of  $\overline{M}$  given by

$$\bar{\mathbf{R}}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]}Z, \quad \text{for all } X, Y, Z \in \mathfrak{X}(\overline{M}).$$

Now, from [15, Prop. 7.42] and using the fact that  $F$  is flat, we have

$$\begin{aligned} \sum_k \bar{g}(\bar{\mathbf{R}}(Y, U_k)U_k, Y) &= (n-1) \frac{f'(\tau)^2}{f(\tau)^2} |Y|^2 - (n-2)(\log f)''(\tau) g(Y, \nabla \tau)^2 \\ &\quad - (\log f)''(\tau) |\nabla \tau|^2 |Y|^2. \end{aligned} \quad (26)$$

From these equations, taking into account the assumptions, we have the Ricci curvature of  $M$  to be non-negative.  $\square$

**Proposition 4** *There is no complete maximal hypersurface  $M$  in a spatially open Robertson-Walker spacetime  $\overline{M} = I \times_f F$  with flat fiber that obeys the Null convergence Condition such that the restriction of the expanding/contracting function  $\text{div}(\partial_t)$  to  $M$  satisfies  $\inf_M |\text{div}(\partial_t)| > 0$ .*

*Proof.* From Lemma 3 we get for any maximal hypersurface  $M$  in  $\overline{M}$

$$\text{Ric}(Y, Y) \geq \frac{n-1}{n^2} \text{div}(\partial_t) \Big|_M^2 |Y|^2, \quad (27)$$

for all  $Y \in \mathfrak{X}(M)$ . Now, using our assumptions and (27) we obtain that the Ricci curvature of  $M$  is bounded from below by a positive constant. If  $M$  is complete the classical Bonnet-Myers Theorem ensures its compactness. However, this contradicts the fact that in a spatially open spacetime there are no compact spacelike hypersurfaces.  $\square$

Proposition 4 enables us to obtain the following non-existence results in some well-known Robertson-Walker spacetimes.

**Corollary 5** *There are no complete maximal hypersurfaces in the  $(n+1)$ -dimensional steady state spacetime  $\mathbb{R} \times_{e^t} \mathbb{R}^n$ .*

**Corollary 6** *There are no complete maximal hypersurfaces bounded away from future infinity in the  $(n+1)$ -dimensional Einstein-de Sitter spacetime  $\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$ .*

This generalizes and improves Rubio's result in [20] to the case of arbitrary dimension. Analogously, we have

**Corollary 7** *There are no complete maximal hypersurfaces bounded away from future infinity in the  $(n+1)$ -dimensional Roberson-Walker Radiation Model spacetime  $\mathbb{R}^+ \times_{(2at)^{1/2}} \mathbb{R}^n$ , with  $a > 0$ .*

## 4 Main result

As a consequence of Lemma 1 and the Liouville-type result given in Lemma 2, we can state our principal result

**Theorem 8** *Let  $\overline{M} = I \times_f F$  be a Robertson-Walker spacetime with flat fiber that obeys the Null Convergence Condition. Then, the only complete maximal hypersurfaces immersed in  $\overline{M}$  satisfying  $\inf \left\{ (n+1) \frac{f'(\tau)^2}{f(\tau)^2} - n \frac{f''(\tau)}{f(\tau)} \right\} > 0$  are the spacelike slices  $\{t_0\} \times F$  with  $f'(t_0) = 0$ .*

*Proof.* From Lemmas 1, 2 and 3 we obtain that the hyperbolic angle must vanish identically on the maximal hypersurface.  $\square$



**Remark 9** Observe that the assumption on the function  $(n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)}$  defined on the hypersurface is scarcely restrictive, even if combined with the NCC. In fact, if we consider its extension  $(n+1)\frac{f'(t)^2}{f(t)^2} - n\frac{f''(t)}{f(t)}$  defined on the spacetime, we have from the NCC that  $(n+1)\frac{f'(t)^2}{f(t)^2} - n\frac{f''(t)}{f(t)} = \frac{f'(t)^2}{f(t)^2} - n(\log f)''(t) \geq 0$ .

However, if we assume that the warping function is defined on the largest possible domain, i.e., it is inextendible, we can find two cases where the required inequality on the infimum does not hold:

1. When both  $f'$  and  $f''$  vanish simultaneously at some point in  $I = ]a, b[$ . This obviously happens in the Lorentz-Minkowski spacetime, where an analogous uniqueness result does not hold. Note that  $\mathbb{L}^{n+1}$  is a vacuum solution. What is more, if there is real presence of matter in the spacetime we can discard this case.
2. If  $\lim_{t \rightarrow b} \frac{f'(t)^2}{f(t)^2} = \lim_{t \rightarrow b} (\log f)''(t) = 0$ . This is the case in the Einstein-de Sitter spacetime. Even more, the inequality will not hold either in the less realistic case where  $\lim_{t \rightarrow a} \frac{f'(t)^2}{f(t)^2} = \lim_{t \rightarrow a} (\log f)''(t) = 0$ .

**Remark 10** Furthermore, this theorem improves some previous uniqueness results for complete maximal hypersurfaces (see [18] and [19], for instance) without making restrictive assumptions on the maximal hypersurface such as having a bounded hyperbolic angle or lying between two spacelike slices. Moreover, we would just need the fiber to be Ricci-flat instead of flat if we knew beforehand that the Ricci curvature of every complete maximal hypersurface in the spacetime is bounded from below. In this way, we could extend our results to GRW spacetimes with Ricci-flat fiber.

We will give now two models where Theorem 8 holds.

**Example 11** Let us consider the Robertson-Walker spacetime  $\overline{M} = \mathbb{R} \times_f \mathbb{R}^n$  with warping function  $f(t) = e^{-t^2}$ . This spacetime obeys NCC, since  $(\log f)''(t) = -2$ . Moreover, any maximal hypersurface immersed in  $\overline{M}$  satisfies

$$\inf \left\{ (n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)} \right\} = \inf \{2n + 4\tau^2\} > 0.$$

Therefore, the only complete maximal hypersurface in  $\overline{M}$  is the spacelike slice  $\{0\} \times \mathbb{R}^n$ .

This spacetime models a relativistic universe without singularities (in the sense of [15, Def. 12.16]) that goes from an expanding phase to a contracting one. The physical space in this transition of phase is represented by the spacelike slice  $\{0\} \times \mathbb{R}^n$ .

**Example 12** We obtain another example of a Robertson-Walker spacetime satisfying the assumptions in Theorem 4 by considering  $\overline{M} = I \times_f \mathbb{R}^n$ . Where  $I = ]-a, a[$  and the warping function is  $f(t) = \sqrt{a^2 - t^2}$ , being  $a$  a positive constant. Let us remark that this spacetime behaves like

the Robertson-Walker model proposed by Friedmann with constant sectional curvature of the fiber equal to one (see [15, Chap. 12]), since it has a big bang singularity at  $t = -a$  as well as a big crunch at  $t = a$  [15, Def. 12.16].

For this spacetime,  $(\log f)''(t) = -\frac{a^2+t^2}{(a^2-t^2)^2} \leq 0$ , so it satisfies NCC. Furthermore, for every maximal hypersurface in  $\overline{M}$

$$\inf \left\{ (n+1) \frac{f'(\tau)^2}{f(\tau)^2} - n \frac{f''(\tau)}{f(\tau)} \right\} = \inf \left\{ \frac{n(a^2 + \tau^2) + \tau^2}{(a^2 - \tau^2)^2} \right\} > 0.$$

Hence, the only complete maximal hypersurface in this spacetime is the spacelike slice  $\{0\} \times \mathbb{R}^n$ , which represents the physical space in the transition from an expanding phase of the spacetime to a contracting one.

**Remark 13** Note that Corollaries 5, 6 and 7 can also be easily deduced from Theorem 8.

## 4.1 Physical meaning of our mathematical assumptions

There is currently great interest in the study of General Relativity in arbitrary dimensions due to several reasons, such as the creation of unified theories or the methodological considerations associated with the possibility of understanding general features for the simpler (2+1)-dimensional models (see [22] and references therein).

We can build a simple cosmological model that will help us to better understand the hypotheses assumed in this article. In order to do so, we will start with our manifold  $\overline{M} = I \times_f \mathbb{R}^n$ , where the lines  $I \times p$  will be the worldlines of the galactic flow. Furthermore, if for each  $p \in \mathbb{R}^n$  we parametrize  $I \times \{p\}$  by  $\gamma_p(t) = (t, p)$  we can define our *galaxies*  $\gamma_p$  as the integral curves of the velocity vector field  $\partial_t$ . In particular, the function  $t$  is the common proper time of all galaxies. By taking  $t$  as a constant, we get the hypersurface

$$M(t) = \{t\} \times \mathbb{R}^n = \{(t, p) : p \in \mathbb{R}^n\}.$$

The distance between two *galaxies*  $\gamma_p$  and  $\gamma_q$  in  $M(t)$  is  $f(t)d(p, q)$ , where  $d$  is the Riemannian distance in the fiber  $\mathbb{R}^n$ . In particular, when  $f$  has positive derivative the spaces  $M(t)$  are expanding. Moreover, if  $f'' > 0$ , Robertson-Walker spacetimes model universes in accelerated expansion.

Some Robertson-Walker spacetimes satisfying the *Null Convergence Condition* can be suitable modified models of gravity. For instance, the steady state spacetime verifies  $f'(t) = e^t > 0$  and so the spaces  $M(t)$  are expanding. In addition, this expansion is accelerated since  $f''(t) = e^t$ . Therefore, the steady state spacetime constitutes an accelerated expanding spacetime.

On the other hand, astronomical evidence indicates that the universe can be modeled (in smoothed, averaged form) as a spacetime containing a perfect fluid whose *molecules* are the galaxies. Classically, the dominant contribution to the energy density of the galactic fluid is the mass of the galaxies, with a smaller pressure due mostly to radiations. Nevertheless, over the 90's, evidences for the most striking result in modern cosmology have been steadily growing, namely the

existence of a cosmological constant which is driving the current acceleration of the universe as first observed in [16], [17]. Different models for dark energy cosmology and their equivalences can be seen in [2]. Note that a positive vacuum energy density resulting from a cosmological constant implies a negative pressure and vice versa.

Thus, it is natural that several exact solutions to the Einstein field equation

$$\overline{\text{Ric}} - \frac{1}{2}\overline{S}\overline{g} = 8\pi T \quad (28)$$

have been obtained by considering a continuous distribution of matter as a perfect fluid.

Recall that a *perfect fluid* (see, for example, [15, Def. 12.4]) on a spacetime  $\overline{M}$  is a triple  $(U, \rho, \mathfrak{p})$  where

1.  $U$  is a timelike future-pointing unit vector field on  $\overline{M}$  called the *flow vector field*.
2.  $\rho, \mathfrak{p} \in C^\infty(\overline{M})$  are, respectively, the *energy density* and the *pressure* functions.
3. The *stress-energy momentum tensor* is

$$T = (\rho + \mathfrak{p})U^{\overline{b}} \otimes U^{\overline{b}} + \mathfrak{p}\overline{g},$$

where  $\overline{g}$  is the metric of the spacetime  $\overline{M}$ .

For an instantaneous observer  $v$ , the quantity  $T(v, v)$  is interpreted as the energy density, i.e., the mass-energy per unit of volume, measured by this observer. For normal matter, this quantity must be non-negative, i.e., the tensor  $T$  must obey the *weak energy condition*. It is easy to see that an exact solution to (28) for a stress-energy tensor which obeys the Weak Energy Condition must satisfy the null energy condition, that is,  $\overline{\text{Ric}}(z, z) \geq 0$  for every null vector  $z$ . Nevertheless, perfect fluids can also be used to model another scenarios of universes at the dark energy dominated stage (see [8]).

In the case of a Robertson-Walker spacetime with flat fiber which is filled with a perfect fluid, the density and pressure functions are given by

$$8\pi\rho = \frac{n(n-1)}{2} \frac{f'^2}{f^2}$$

and

$$8\pi\mathfrak{p} = -(n-1)\frac{f''}{f} - \frac{(n-1)(n-2)}{2} \frac{f'^2}{f^2},$$

being  $f$  the warping function. For this family of classical models with positive density and non-negative pressure ( $\rho > 0, \mathfrak{p} \geq 0$ ), every maximal hypersurface included in a region of the spacetime in which the stress-energy momentum tensor is far from zero satisfies the condition  $\inf \left\{ (n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)} \right\} > 0$ , which is equivalent in our model to  $\inf \left\{ \frac{8\pi}{n-1} \left( \frac{n^2+2}{n}\rho + n\mathfrak{p} \right) \right\} > 0$ . For instance, this happens near a physical singularity.

On the other hand, as the example of the steady state spacetime shows, the condition

$\inf \left\{ (n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)} \right\} > 0$  holds in certain models with negative pressure.

Finally, we may wonder whether our assumptions in Theorem 8 are compatible with other usual energy conditions [10, Sec. 4.3]. In order to better understand these conditions, we will express them for a perfect fluid in terms of the energy density and pressure functions as well as using the warping function  $f$  of the Robertson-Walker spacetime with flat fiber. Hence, for a Robertson-Walker spacetime with flat fiber filled with a perfect fluid we have

- The *Weak Energy Condition* implies that  $\rho \geq 0$  and  $\rho + \mathfrak{p} \geq 0$ ; which is equivalent to  $(\log f)'' \leq 0$ , since in our perfect fluid model the energy density is always non-negative.
- The *Strong Energy Condition* stipulates that  $\rho + \mathfrak{p} \geq 0$  and  $\rho + n\mathfrak{p} \geq 0$ ; or equivalently,  $(\log f)'' \leq 0$  and  $\frac{n-3}{2} \frac{f'^2}{f^2} + \frac{f''}{f} \leq 0$  (for  $n = 3$ , this last equation leads to  $f'' \leq 0$ , which is equivalent to the *Timelike Convergence Condition* in this case).
- The *Dominant Energy Condition* is written as  $\rho \geq |\mathfrak{p}|$ , which for positive pressure can be written as  $(n-1) \frac{f'^2}{f^2} + \frac{f''}{f} \geq 0$  and for negative pressure implies  $(\log f)'' \leq 0$ .

Thus, our assumptions can be compatible in many cases with these energy conditions.

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